MULTIPLIERS ON D_α[1]

BY
GERALD D. TAYLOR

In this paper we study multiplication operators on certain Hilbert spaces of vector-valued functions.

Let H be a given Hilbert space with inner product $\langle\ \rangle$. Fix α a real number, and set

$$D_\alpha = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \,\middle|\, a_n \in H \text{ for } n = 0,1,2,\cdots, \text{ and } \sum_{n=0}^{\infty} (n+1)^\alpha \| a_n \|_H^2 < \infty \right\}.$$

D_α is a Hilbert space with inner product $(f,g) = \sum (n+1)^\alpha \langle a_n, b_n \rangle$ where $f(z) = \sum a_n z^n$, $g(z) = \sum b_n z^n$ both belong to D_α and the absence of indices on the summation signs will henceforth indicate the sum is from 0 to ∞. The functions of D_α are analytic vector-valued functions mapping the open unit disc into H. Further note that for $a \in H$ the constant function $f_a \equiv a$ is in D_α.

Let λ_z^α denote the transformation which maps D_α into H by $\lambda_z^\alpha(f) = f(z)$ for each $f \in D_\alpha$ and z a complex number of modulus less than 1. In §I, we show that λ_z^α is a bounded linear transformation with norm $(\sum (n+1)^{-\alpha} |z|^{2n})^{1/2}$.

For $\alpha \leqq 0$ the norm of D_α may also be given by an integral. Lemma 2 shows that the norm of D_α is equivalent to

$$\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \| f(re^{i\theta}) \|_H^2 (1 - r^2)^{-\alpha - 1} r \, dr \, d\theta,$$

when $\alpha < 0$. For $\alpha = 0$ we have the well-known Hardy space of square-summable functions [3] and the norm is equivalent to

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \| f(re^{i\theta}) \|_H^2 \, d\theta.$$

The symbols $L(H,H)$ and $L(D_\alpha, D_\beta)$ shall denote the algebras of all bounded linear transformations of H into H and D_α into D_β, respectively.

DEFINITION 1. Let $h(z)$ be an operator-valued function mapping the open unit disc into $L(H,H)$. Then $h(z)$ is a multiplier from D_α to D_β if $h \cdot f \in D_\beta$ for each $f \in D_\alpha$, where $h \cdot f$ denotes pointwise multiplication of the two functions.

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Let $M(D_\alpha, D_\beta)$ denote the set of all multipliers from D_α to D_β. The closed graph theorem implies that T_h mapping D_α into D_β by $T_h(f) = h\cdot f$ for h a multiplier and $f\in D_\alpha$ is a bounded linear transformation.

I. **Characterizations.** We begin by proving a few elementary facts about D_α.

LEMMA 1. *λ_z^α is a bounded linear transformation with norm*

$$(\Sigma(n+1)^{-\alpha}|z|^{2n})^{1/2}.$$

Proof. Let $f(z) = \Sigma a_n z^n \in D_\alpha$, then

$$\|\lambda_z^\alpha(f)\|_H^2 = \|f(z)\|_H^2 \leqq \left(\sum_{n=0}^{\infty} \|a_n\|_H |z|^n\right)^2$$

$$\leqq \left(\sum_{n=0}^{\infty}(n+1)^\alpha \|a_n\|_H^2\right)\left(\sum_{n=0}^{\infty}(n+1)^{-\alpha}|z|^{2n}\right)$$

by the Cauchy-Schwartz inequality. Thus $\|\lambda_z^\alpha\| \leqq (\Sigma(n+1)^{-\alpha}|z|^{2n})^{1/2}$. To show that this is equality, fix z ($|z|<1$), let $a\in H$ be of norm 1 and set $f_{a,z}(w) = \Sigma(n+1)^{-\alpha}a(\bar{z}w)^n$. Note $f_{a,z}\in D_\alpha$ since

$$\|f_{a,z}\|_\alpha^2 = \sum_{n=0}^{\infty}(n+1)^{-\alpha}|z|^{2n} < \infty.$$

Thus

$$\|\lambda_z^\alpha(f_{a,z})\|_H = \left\|\sum_{n=0}^{\infty}(n+1)^{-\alpha}|z|^{2n}a\right\|_H$$

$$= \sum_{n=0}^{\infty}(n+1)^{-\alpha}|z|^{2n}$$

$$= \left(\sum_{n=0}^{\infty}(n+1)^{-\alpha}|z|^{2n}\right)^{1/2} \|f_{a,z}\|_\alpha.$$

LEMMA 2. *For $\alpha<0$, the norm of D_α is equivalent to*

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi}\|f(re^{i\theta})\|_H^2(1-r^2)^{-1-\alpha}\,r\,dr\,d\theta.$$

Proof. We begin by noting that

$$\|f(z)\|_H^2 = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} z^k\bar{z}^m\langle a_k, a_m\rangle$$

for $f(z) = \Sigma a_n z^n \in D_\alpha$ and z of modulus less than 1.

Thus

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi}\|f(re^{i\theta})\|_H^2(1-r^2)^{-1-\alpha}r\,dr\,d\theta$$

$$=2\sum_{k=0}^{\infty}\|a_k\|_H^2\int_0^1 r^{2k+1}(1-r^2)^{-1-\alpha}dr$$

$$=2\sum_{k=0}^{\infty}\frac{\|a_k\|_H^2}{(-\alpha)\binom{k-\alpha}{\alpha}}$$

where we have integrated by parts k times. From [1, Chapter 5, §4] one obtains that the above series is asymptotic to a constant times $\Sigma(n+1)^\alpha\|a_n\|_H^2$ and the lemma follows.

LEMMA 3. *T_h for $h\in M(D_\alpha,D_\beta)$ is a bounded linear transformation mapping D_α into D_β.*

Proof. That T_h is linear is obvious. We show that T_h is bounded by applying the closed graph theorem. Let $\{f_n\}_{n=1}^{\infty}\in D_\alpha$ and $f_n\to f$ in D_α. Also let $h\cdot f_n=g_n$ for all n and $g_n\to g$ in D_β. We must show $h\cdot f=g$ since $f\in D_\alpha$ by completeness and T_h has all of D_α as its domain. Fix z such that $|z|<1$, then

$$g(z)=\lambda_z^\beta(g)=\lim_{n\to\infty}\lambda_z^\beta(h\cdot f_n)=h(z)\lim_{n\to\infty}\lambda_z^\alpha(f_n)=h(z)f(z),$$

using the fact that λ_z^β is continuous (Lemma 1).

We now give a necessary condition for an operator-valued function to belong to $M(D_\alpha,D_\beta)$. In order to obtain this condition we need the following lemma.

LEMMA 4. *Let $h\in M(D_\alpha,D_\beta)$, let k be a nonnegative integer and let z_0 be a complex number ($|z_0|<1$). Then there exists an operator $U_k(z_0)\in L(H,H)$ such that*

$$\left.\frac{d^k(h(z)\cdot f_a)}{dz^k}\right|_{z=z_0}=U_k(z_0)\cdot a,$$

where $a\in H$ and $f_a\in D_\alpha$ with $f_a\equiv a$. Note that $U_0(z_0)=h(z_0)$.

Proof. Let $a\in H$ and set $f_a(z)\equiv a$. Then f_a is a constant function in D_α and will be denoted by a throughout the paper. The proof will be by induction on k. Assume the theorem is true for $k<n$. Fix z_0 of modulus less than 1 and observe that

$$\left.\frac{d^n(h(z)\cdot a)}{dz^n}\right|_{z=z_0}=\frac{d}{dz}\left.\left(\frac{d^{n-1}(h(z)\cdot a)}{dz^{n-1}}\right)\right|_{z=z_0}$$

$$=\lim_{t\to 0}\frac{U_{n-1}(z_0+t)-U_{n-1}(z_0)}{t}\cdot a,$$

where the limit is evaluated in the norm of H. Let

$$U_{n-1}(z_0,t) = \frac{U_{n-1}(z_0+t) - U(z_0)}{t}.$$

Note that $U_{n-1}(z_0,t) \in L(H,H)$ for t sufficiently small by hypothesis. Also the analyticity of the functions of D_β implies

$$\left.\frac{d^n(h(z)\cdot a)}{dz^n}\right|_{z=z_0} = \lim_{t\to 0} U_{n-1}(z_0,t)\cdot a$$

is a fixed vector in H. Thus the family $\{U_{n-1}(z_0,t)\}_{t\in S}$, where S is a disc about z_0 in the complex plane with radius smaller than $1-|z_0|$, is uniformly bounded by the uniform boundedness principle. Thus by the uniform boundedness principle stated in a different manner [2, Theorem 2. 12. 1, p. 50] if follows that that $U_n(z_0)$ defined by $U_n(z_0)\cdot a = \lim_{t\to 0} U_n(z_0,t)\cdot a$ for each $a\in H$ belongs to $L(H,H)$.

THEOREM 1. *Let $h \in M(D_\alpha, D_\beta)$. Then h is analytic (given by a Taylor series with coefficients in $L(H,H)$) and*

$$\| h(z) \|_L \leqq \| T_h \| \frac{\| \lambda_z^\beta \|}{\| \lambda_z^\alpha \|}$$

for each z of modulus less than 1. By $\| A \|_L$ we mean the norm of A in $L(H,H)$.

Proof. Fix $a \in D_\alpha$. Then $h(z)\cdot a = \Sigma b_n z^n \in D_\beta$, where $b_n \in H$ for $n = 0,1,2,\cdots$,

$$b_n = \frac{1}{n!}\left.\frac{d^n(h(z)\cdot a)}{dz^n}\right|_{z=0}.$$

By Lemma 4, $b_n = (1/n!)\,U_n(0)\cdot a$ where $U_n(0)\in L(H,H)$. Thus fixing z_0 such that $|z_0|<1$, we have that $h(z_0)\cdot a = (\Sigma(1/n!)\,U_n(0)z_0^n)\cdot a$ for each $a\in H$. Since both $h(z_0)$ and $\Sigma(1/n!)\,U_n(0)z_0^n$ belong to $L(H,H)$ it follows that they are equal. Finally, it is clear that $\Sigma(1/n!)\,U_n(0)z^n$ has a radius of convergence of at least 1 as $\|(1/n!)\,U_n(0)\cdot a\|_H \leqq (n+1)^{-\beta/2}\|T_h\|\;\|a\|_\alpha$ for $a\in H$ implies

$$\left\| \frac{1}{n!} U_n(0) \right\|_L \leqq M(n+1)^{-\beta/2}$$

by the uniform boundedness principle.

To obtain the second part of the theorem we shall need the following special function. Let $a\in H$ be of norm 1 and w be a complex number of modulus less than 1. Define $f_{a,w}(z)\in D_\alpha$ by $f_{a,w}(z) = \Sigma(n+1)^{-\alpha}a(\bar w z)^n$. Note that

$$\|f_{a,w}\|_\alpha^2 = \Sigma(n+1)^{-\alpha}|w|^{2n} \text{ and } \|f_{a,w}(w)\|_H = \Sigma(n+1)^{-\alpha}|w|^{2n}.$$

Fix w of modulus less than 1 and let $\varepsilon>0$ be given. Choose $a\in H$ such that $\|a\|_H = 1$ and $\|h(w)\|_L < \|h(w)\cdot a\|_H + \varepsilon$. Thus

$$\| h(w) \|_L \, \|f_{a,w}(w)\|_H < (\| h(w)\cdot a \|_H + \varepsilon)(\Sigma(n+1)^{-\alpha} | w |^{2n})$$
$$\leqq \| \lambda_w^\beta \| \; \| T_h \| \; \|f_{a,w}\|_\alpha + \varepsilon \, \Sigma(n+1)^{-\alpha} | w |^{2n}.$$

Since $\|f_{a,w}(w)\|_H$ is nonzero, we may divide both sides by it obtaining

$$\| h(w) \|_L < \| T_h \| \frac{\| \lambda_w^\beta \|}{\| \lambda_w^\alpha \|} + \varepsilon.$$

Letting $\varepsilon \to 0$, the theorem follows.

We now turn to the task of obtaining sufficient conditions that an analytic operator-valued function be a multiplier for different choices of α and β. The first case we shall consider is $0 > \alpha \geqq \beta$.

THEOREM 2. *$h \in M(D_\alpha, D_\beta)$ for $0 > \alpha \geqq \beta$ if and only if $h(z)$ is an analytic operator-valued function mapping the unit disc into $L(H,H)$ and $\| h(z) \|_L = O((1 - |z|^2)^{(\beta-\alpha)/2})$.*

Proof. For $h \in M(D_\alpha, D_\beta)$, Theorem 1 implies h is analytic and

$$\| h(z) \|_L \leqq \| T_h \| \frac{\| \lambda_z^\beta \|}{\| \lambda_z^\alpha \|} \leqq K(1 - |z|^2)^{(\beta-\alpha)/2} ,$$

since $\Sigma(n+1)^{-\beta} |z|^{2n}$ is asymptotic to $(1-|z|^2)^{\beta-1}$ for $1 > \beta$ [1, Chapter 5, p. 96 and p.108].

Conversely, if h is an analytic operator-valued function mapping the unit disc into $L(H,H)$ and $\| h(z) \|_L \leqq C(1 - |z|^2)^{(\beta-\alpha)/2}$, then by the integral representation of the norm of D_α for $\alpha < 0$ it follows that $h \in M(D_\alpha, D_\beta)$.

For $0 > \alpha = \beta$, $M(D_\alpha, D_\alpha)$ consists of bounded analytic operator-valued functions. It is well known that $M(D_0, D_0)$ is also precisely the bounded analytic operator-valued functions.

Next we consider the case where $\beta > \alpha$. In this case a convexity theorem, similar to the Riesz-Thorin Convexity Theorem [4, Chapter 12, p. 93], is needed. We shall simply state this theorem since the proof is similar to that of the Riesz-Thorin Convexity Theorem.

THEOREM 3. *If $h \in M(D_{\alpha_1}, D_{\beta_1})$ and $h \in M(D_{\alpha_2}, D_{\beta_2})$ and $\alpha = (1-\lambda)\alpha_1 + \lambda\alpha_2$, $\beta = (1-\lambda)\beta_1 + \lambda\beta_2$, $0 \leqq \lambda \leqq 1$, then $h \in M(D_\alpha, D_\beta)$ and $\| T_h \|_{\alpha,\beta} \leqq \| T_h \|_{\alpha_1,\beta_1}^{1-\lambda} \| T_h \|_{\alpha_2,\beta_2}^{\lambda}$.*

COROLLARY 1. *$M(D_\alpha, D_\alpha) \subset M(D_\beta, D_\beta)$ for $\alpha > \beta$.*

Proof. If $\beta < 0$ and $h \in M(D_\alpha, D_\alpha)$ then Theorem 2 implies $h \in M(D_\beta, D_\beta)$. For $\beta \geqq 0$ and $h \in M(D_\alpha, D_\alpha)$, Theorems 1 and 2 imply that $h \in M(D_0, D_0)$. Thus the corollary follows immediately from Theorem 3.

THEOREM 4. *$M(D_\alpha, D_\beta) = \{h(z) \,|\, h(z) \equiv 0\}$ for $\beta > \alpha$.*

Proof. The proof will be given in three parts, namely: (i) $\beta > 1 \geqq \alpha$, (ii) $1 \geqq \beta > \alpha$ and (iii) $\beta > \alpha > 1$. Theorem 1 gives the inequality

$$\| h(z) \|_L \leqq \| T_h \| \left[\frac{\sum_{n=0}^{\infty} (n+1)^{-\beta} |z|^{2n}}{\sum_{n=0}^{\infty} (n+1)^{-\alpha} |z|^{2n}} \right]^{1/2}.$$

For (i) we note that as $|z| \to 1$ the series in the numerator approaches a constant by Abel's Theorem and the series in the denominator tends to infinity. This implies that $\| h(z) \|_L$ approaches zero as $|z| \to 1$, so by the maximum modulus theorem for analytic vector-valued functions [2, Chapter 3, p. 59] it follows that $h(z) \equiv 0$.

In case (ii) we may assume that $1 > \beta$ (for $\beta = 1$ we replace it by $(\beta + \alpha)/2$ strengthening the inequality). Here as in Theorem 2, we note that

$$\| h(z) \|_L \leqq C(1 - |z|^2)^{(\beta - \alpha)/2},$$

C a fixed constant independent of z. Letting $|z| \to 1$, we see that $\| h(z) \|_L \to 0$ since $\beta > \alpha$.

Finally in case (iii) we note that both series converge as $|z| \to 1$. Thus $h(z)$ is a bounded analytic operator-valued function. By Theorem 2, $h \in M(D_{-2}, D_{-2})$. Hence Theorem 3 implies $h \in M(D_{\alpha'}, D_{\beta'})$ where $\alpha' = (1 - \lambda)(-2) + \lambda\alpha$ and $\beta' = (1 - \lambda)(-2) + \lambda\beta$. Let $\lambda = 2/(2 + \alpha)$, then $\alpha' = 0$ and $\beta' > 0$. Therefore $h(z) \equiv 0$ by either (i) or (ii), depending upon β'.

For H infinite dimensional and the remaining choices of α and β a necessary and sufficient condition for an analytic operator-valued function to belong to $M(D_\alpha, D_\beta)$ is not known. In the special case of H finite dimensional, $\alpha > 1$ and $\alpha > \beta$ a necessary and sufficient condition will be given. We shall consider the remaining values of α and β in three parts. Namely, (i) $1 \geqq \alpha \geqq \beta \geqq 0$, (ii) $1 \geqq \alpha \geqq 0 \geqq \beta$, and (iii) $\alpha > 1$, $\alpha \geqq \beta$. Note that two of these sufficient conditions contain the case $\alpha = \beta = 0$ which has already been characterized and each will say that the function must have absolutely convergent Taylor coefficients which is not as good as the known sufficient condition on $M(D_0, D_0)$. Due to the similarity of the proofs of each case we shall prove only one case and state the others.

Theorem 5. *Let $\{n_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of nonnegative integers. Let $\phi(n_k) \geqq 1$,*

$$\sum_{k=0}^{\infty} \frac{1}{(n_k + 1)^\alpha \phi(n_k)} = C_1 < \infty, \qquad 1 \geqq \alpha \geqq \beta \geqq 0,$$

and

$$\sum_{p=0}^{k} \frac{1}{(n_k - n_p + 1)^\alpha \phi(n_p)} \leqq C_2 < \infty$$

where C_2 is independent of k. If

$$h(z) = \sum_{k=0}^{\infty} A_{n_k} z^{n_k} \text{ and } \sum_{k=0}^{\infty} (n_k + 1)^\beta \phi(n_k) \| A_{n_k} \|_L^2 < \infty$$

then $h \in M(D_\alpha, D_\beta)$.

Proof. Let $f(z) = \sum b_n z^n \in D_\alpha$ and form

$$\| h \cdot f \|_\beta^2 = \sum_{k=0}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} (n+1)^\beta \left\| \sum_{p=0}^{k} A_{n_p} b_{n-n_p} \right\|_H^2$$

$$\leqq \sum_{k=0}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} (n+1)^\beta \left(\sum_{p=0}^{k} \frac{1}{(n_p+1)^\beta \phi(n_p)(n-n_p+1)^\alpha} \right)$$

$$\times \left(\sum_{p=0}^{k} (n_p+1)^\beta \phi(n_p) \| A_{n_p} \|_L^2 (n-n_p+1)^\alpha \| b_{n-n_p} \|_H^2 \right).$$

Now for $n_k \leqq n < n_{k+1}$,

$$(n+1)^\beta \sum_{p=0}^{k} \frac{1}{(n_p+1)^\beta \phi(n_p)(n-n_p+1)^\alpha}$$

$$\leqq \left(\frac{n+1}{n+2} \right)^\beta \sum_{p=0}^{k} \frac{1}{(n_k-n_p+1)^{\alpha-\beta} \phi(n_p)} \left(\frac{1}{n_p+1} + \frac{1}{n_k-n_p+1} \right)^\beta$$

$$\leqq \sum_{p=0}^{k} \frac{1}{(n_k-n_p+1)^{\alpha-\beta}(n_p+1)^\beta \phi(n_p)} + \sum_{p=0}^{k} \frac{1}{(n_k-n_p+1)^\alpha \phi(n_p)}$$

$$\leqq \left(\sum_{p=0}^{k} \frac{1}{(n_p+1)^\alpha \phi(n_p)} \right)^{\beta/\alpha} \left(\sum_{p=0}^{k} \frac{1}{(n_k-n_p+1)^\alpha \phi(n_p)} \right)^{(\alpha-\beta)/\alpha} + C_2$$

$$\leqq C_1^{\beta/\alpha} C_2^{(\alpha-\beta)/\alpha} + C_2 = C < \infty .$$

Thus

$$\| h \cdot f \|_\beta^2 \leqq C \sum_{k=0}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} \sum_{p=0}^{k} (n_p+1)^\beta \phi(n_p) \| A_{n_p} \|_L^2 (n-n_p+1)^\alpha \| b_{n-n_p} \|_H^2$$

$$\leqq C \left(\sum_{k=0}^{\infty} (n_k+1)^\beta \phi(n_k) \| A_{n_k} \|_L^2 \right) \| f \|_\alpha^2$$

$$= A \| f \|_\alpha^2 .$$

THEOREM 6. *Let* $\{n_k\}_{k=0}^{\infty}$ *be a strictly increasing sequence of nonnegative integers. Let* $\phi(n_k) \geqq 1$, $1 \geqq \alpha \geqq 0 \geqq \beta$, *and*

$$\sum_{p=0}^{k} \frac{1}{(n_k-n_p+1)^\alpha \phi(n_p)} \leqq C_1$$

where C_1 *is a positive constant independent of* k. *If*

$$h(z) = \sum_{k=0}^{\infty} A_{n_k} z^{n_k} \text{ and } \sum_{k=0}^{\infty} (n_k + 1)^{\beta} \phi(n_k) \| A_{n_k} \|_L^2 < \infty$$

then $h \in M(D_\alpha, D_\beta)$.

THEOREM 7. *Let* $\alpha > 1, \alpha \geqq \beta$ *and* $h(z) = \sum_{n=0}^{\infty} A_n z^n$. *If* $\sum_{n=0}^{\infty} (n+1)^{\beta} \| A_n \|_L^2 < \infty$ *then* $h \in M(D_\alpha, D_\beta)$.

We now consider the case where H is finite dimensional and show that in this case the converse of Theorem 7 is valid. We shall also give an example of a multiplier when H is infinite dimensional for which the converse of Theorem 7 is not true.

Let H be an m-dimensional Hilbert space with basis $\{e_i\}_{i=1}^m$ and inner product $\langle\ \rangle$. Here $L(H,H)$ consists of all $m \times m$ matrices over the complex numbers. Note that for $A = (a_{ij})_{i,j=1}^m \in L(H,H)$,

$$\| A \|_L \leqq \sum_{i=1}^m \left(\sum_{j=1}^m | a_{ij} |^2 \right)^{1/2}.$$

THEOREM 8. *Let H be an m-dimensional Hilbert space and*

$h(z) = \sum A_n z^n \in M(D_\alpha, D_\beta)$ *where* $A_n = (a_{ij}^n)_{i,j=1}^m$ *then* $\sum (n+1)^{\beta} \| A_n \|_L^2 < \infty$.

Proof. We assume $\alpha > \beta$ for in the light of Theorem 4, the other case is vacuous. First note that

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)^{\beta} \| A_n \|_L^2 &\leqq \sum_{n=0}^{\infty} (n+1)^{\beta} \left\{ \sum_{i=1}^m \left(\sum_{j=1}^m | a_{ij}^n |^2 \right)^{1/2} \right\}^2 \\ &\leqq 2^{m-1} \sum_{n=0}^{\infty} (n+1)^{\beta} \left(\sum_{i=1}^m \sum_{j=1}^m | a_{ij}^n |^2 \right). \end{aligned}$$

Let $b = \sum \gamma_i e_i \in H$ *be* of norm 1, then

$$\| T_h \|^2 \geqq \| T_h b \|_\beta^2 = \sum_{n=0}^{\infty} (n+1)^{\beta} \sum_{i=1}^m \left| \sum_{j=1}^m a_{ij}^n \gamma_j \right|^2.$$

By choosing $\gamma_j = 0$ for $j \neq p$ and $\gamma_p = 1$, it follows that

$$\| T_h \|^2 \geqq \sum_{n=0}^{\infty} (n+1)^{\beta} \sum_{i=1}^m | a_{ip}^n |^2$$

for $p = 1, 2, \cdots, m$. Thus

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)^{\beta} \| A_n \|_L^2 &\leqq 2^{m-1} \sum_{n=0}^{\infty} (n+1)^{\beta} \sum_{i=1}^m \sum_{j=1}^m | a_{ij}^n |^2 \\ &= 2^{m-1} \sum_{j=1}^m \sum_{n=0}^{\infty} (n+1)^{\beta} \sum_{i=1}^m | a_{ij}^n |^2 \\ &\leqq m \cdot 2^{m-1} \| T_h \|^2. \end{aligned}$$

Let H be an infinite dimensional Hilbert space with basis $\{e_\gamma\}_{\gamma \in A}$ and let $S = \{e_i\}_{i=0}^{\infty}$ be some subset of this basis such that $\langle e_i, e_j \rangle = \delta_{ij}$. For

$$g(z) = \Sigma a_n z^n \in D_\alpha, \ \| g \|_\alpha^2 = \sum_{n=0}^{\infty} \sum_{\gamma \in A} (n+1)^\alpha |\langle a_n, e_\gamma \rangle|^2.$$

Let $h(z) = \Sigma (n+1)^{-\beta/2} P_n z^n$ where P_n denotes the orthogonal projection of H into the subspace spanned by e_n of S. For $g(z) = \Sigma b_n z^n \in D_\alpha$ $(\alpha > 1, \alpha \geqq \beta)$,

$$\begin{aligned} \| h \cdot g \|_\beta^2 &= \sum_{n=0}^{\infty} (n+1)^\beta \Big\| \sum_{k=0}^{n} (k+1)^{-\beta/2} P_k b_{n-k} \Big\|_H^2 \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (n+k+1)^\beta (k+1)^{-\beta} |\langle b_n, e_k \rangle|^2 \\ &\leqq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (n+1)^\alpha |\langle b_n, e_k \rangle|^2 \\ &\leqq \sum_{n=0}^{\infty} \sum_{\gamma \in A} (n+1)^\alpha |\langle b_n, e_\gamma \rangle|^2 \\ &= \| g \|_\alpha^2 . \end{aligned}$$

This implies that $h(z)$ is a multiplier from D_α to D_β. Finally, note that

$$\sum_{n=0}^{\infty} (n+1)^\beta \| (n+1)^{-\beta/2} P_n \|_L^2 = \sum_{n=0}^{\infty} 1 = \infty .$$

II. **Examples.** We now give two examples in the case where H is a one-dimensional Hilbert space (i.e. the complex numbers). The first example will be a function, h, such that $h \in D_1$ and $h \notin M(D_1, D_1)$ and the second will be of a multiplier that will imply $M(D_\alpha, D_\alpha) \subset M(D_\beta, D_\beta)$ properly for $\alpha > \beta \geqq 0$.

THEOREM 9. *There exists a function $h(z) = \Sigma_{n=0}^{\infty} a_n z^n$ such that $\Sigma_{n=0}^{\infty} a_n < \infty$, $a_n > 0$ for all n, $a_n \downarrow 0$ and $h \notin M(D_1, D_1)$.*

REMARK. $\Sigma a_n < \infty$ and $a_n \downarrow 0$ imply $(n+1) a_n \to 0$. Thus $\Sigma (n+1) |a_n|^2 \leqq M \Sigma a_n < \infty$ and one sees that $h \in D_1$. For H one-dimensional Theorem 7 becomes $M(D_\alpha, D_\alpha) = D_\alpha$ for $\alpha > 1$. This example shows that $M(D_1, D_1) \neq D_1$. A second way to observe this fact is to note that D_1 contains unbounded functions (i.e. $\Sigma_3^\infty (z^n / n \ln n)$) which by Theorem 1 can not belong to $M(D_1, D_1)$. This reasoning may also be used to show that $M(D_\alpha, D_\alpha) \neq D_\alpha$ for $\alpha < 1$.

Proof. Let $n_2 = 2$, $\gamma > 1$ and $n_k = [\exp(k^2 \ln^\gamma k)] + n_{k-1} + 1$ for $k = 3, 4, 5, \cdots$, where the brackets denote the greatest integer. Let $\varepsilon > 0$ be given and set

$$a_n = \frac{k + \varepsilon / 2^n}{(n_k + 1) \ln(n_k - n_{k-1} - 1)}$$

for $n_{k-1}+1 \leqq n \leqq n_k$ and also set $a_0 > a_1 > a_2 > a_3$ where a_3 will be given by the above formula. We shall first note four facts.

(i)
$$\sum_{n=0}^{\infty} a_n \leqq 3a_0 + \sum_{k=3}^{\infty} \sum_{n=n_{k-1}+1}^{n_k} a_n$$
$$\leqq 3a_0 + \varepsilon + \sum_{k=3}^{\infty} \frac{k(n_k - n_{k-1})}{(n_k+1)\ln(n_k - n_{k-1} - 1)}$$
$$\leqq 3a_0 + \varepsilon + \sum_{k=3}^{\infty} \frac{2}{k \ln^{\gamma} k} < \infty$$

as $\gamma > 1$.

(ii)
$$\sum_{k=3}^{\infty} \frac{k^2}{\ln^{5/4}(n_k - n_{k-1} - 1)} \geqq \sum_{k=3}^{\infty} \frac{k^2}{\ln^{5/4}\{\exp(k^2 \ln^{\gamma} k)\}}$$
$$= \sum_{k=3}^{\infty} \frac{1}{k^{1/2} \ln^{5\gamma/4} k} = \infty .$$

(iii) $n_{k-1}/n_k \to 0$ as $k \to \infty$, this is easily seen since $n_p = O\{\exp(p^2 \ln^{\gamma} p)\}$.

(iv) $a_n \downarrow 0$. We must check only at the jumps since it is clear that $a_n \downarrow 0$ for $n_{k-1}+1 \leqq n \leqq n_k$. Now

$$\frac{a_{n_k}}{a_{n_{k-1}}} = \frac{(n_{k-1}+1)\ln(n_{k-1} - n_{k-2} - 1)(k + e/2^n)}{(n_k+1)\ln(n_k - n_{k-1} - 1)(k - 1 + e/2^{n-1})}$$
$$\leqq \frac{(n_{k-1}+1)(k-1)^2 \ln^{\gamma}(k-1)\cdot 2k}{(n_k+1)\frac{1}{2}k^2 \ln^{\gamma} k \cdot (k-1)}$$
$$= \frac{4(n_{k-1}+1)(k-1)\ln^{\gamma}(k-1)}{(n_k+1)k \ln^{\gamma} k} \downarrow 0$$

as $k \to \infty$.

Let $h(z) = \Sigma a_n z^n$, $f(z) = \Sigma b_n z^n$ where $b_n = 1/(n+e)\ln^{5/8}(n+e)$: Note that $f \in D_1$. Set $h(z)f(z) = \Sigma c_n z^n$. Let $r = 0, 1, 2, \cdots, [(n_k - n_{k-1} - 1)/2]$, then

$$c_{n_k - r} \geqq \sum_{n=0}^{n_k - n_{k-1} - r - 1} a_{n_k - r - n} b_n$$
$$\geqq \frac{k}{(n_k+1)\ln(n_k - n_{k-1} - 1)} \sum_{n=0}^{n_k - n_{k-1} - r - 1} b_n$$
$$\geqq \frac{\frac{1}{2}k \ln^{3/8}(n_k - n_{k-1} - r - 1)}{(n_k+1)\ln(n_k - n_{k-1} - 1)} .$$

Thus

$$\|h\cdot f\|_1^2 \geqq \frac{1}{4}\sum_{k=3}^{\infty}\sum_{r=0}^{[(n_k-n_{k-1}-1)/2]}\frac{(n_k-r+1)k^2\ln^{3/4}(n_k-n_{k-1}-r-1)}{(n_k+1)^2\ln^2(n_k-n_{k-1}-1)}$$

$$\geqq \frac{1}{4}\sum_{k=L}^{\infty}\frac{\frac{1}{2}(n_k+n_{k-1})\frac{1}{2}(n_k-n_{k-1}-3)k^2\ln^{3/4}\frac{1}{2}(n_k-n_{k-1}-1)}{(n_k+1)^2\ln^2(n_k-n_{k-1}-1)}$$

$$\geqq C\sum_{k=3}^{\infty}\frac{k^2}{\ln^{5/4}(n_k-n_{k-1}-1)} = \infty,$$

where C is a positive constant. Thus $h \notin M(D_1, D_1)$.

THEOREM 10. *Fix α and β such that $0<\alpha<\beta\leqq 1$. Then there exists a function, h, such that $h\in M(D_\alpha,D_\alpha)$ and $h\notin M(D_\beta,D_\beta)$.*

REMARK. For H a one-dimensional Hilbert space, Theorem 7 implies $M(D_\alpha,D_\alpha)=D_\alpha$ for $\alpha>1$. Combining this with observation that $D_\alpha\subset D_\beta$ properly for $\alpha>\beta$, it is clear that $M(D_\alpha,D_\alpha)\subset M(D_\beta,D_\beta)$ properly for $\alpha>\beta>1$. Theorem 10 extends this to $\alpha>\beta\geqq 0$.

Proof. Let $n_k=(k+1)^p$ where $p>4/(\beta-\alpha)$, $\phi(n_k)=(k+1)^2$ and

$$h(z) = \sum_{k=0}^{\infty}\frac{z^{n_k}}{(n_k+1)^{\beta/2}}.$$

Note that

$$\sum_{k=0}^{\infty}\frac{1}{(n_k+1)^\alpha\phi(n_k)} \leqq \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} < \infty,$$

and

$$\sum_{p=0}^{k}\frac{1}{(n_k-n_p+1)^\alpha\phi(n_p)} \leqq \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} < \infty,$$

for all k. Also

$$\sum_{k=0}^{\infty}(n_k+1)^\alpha\phi(n_k)\left|\frac{1}{(n_k+1)^{\beta/2}}\right|^2 \leqq \sum_{k=0}^{\infty}\frac{\phi(n_k)}{n_k^{\beta-\alpha}} = \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} < \infty.$$

By Theorem 5, $h\in M(D_\alpha,D_\alpha)$. Finally,

$$\|h\|_\beta^2 = \sum_{k=0}^{\infty}(n_k+1)^\beta\left|\frac{1}{(n_k+1)^{\beta/2}}\right|^2 = \sum_{k=0}^{\infty}1 = \infty.$$

This shows that $h\notin M(D_\beta,D_\beta)$ since $g(z)=\sum_{n=0}^{\infty}A_nz^n\in M(D_\beta,D_\beta)$ implies $\sum_{n=0}^{\infty}(n+1)^\beta\|A_n\cdot a\|_H^2<\infty$ for each vector $a\in H$.

Note that these are also examples regardless of the dimension of H, since any function of the scalar case can make a function of the vector case simply by taking it to be the coefficient of a vector of H or the identity of $L(H,H)$.

III. **Summary.** We have given necessary and sufficient conditions for $0 \geqq \alpha \geqq \beta$ and $\beta > \alpha$. When H is finite dimensional a complete characterization is also given for $\alpha > 1$ and $\alpha \geqq \beta$. Aside from the obvious desire to complete the description of $M(D_\alpha, D_\beta)$ is the general case, the most interesting question left open is probably the lack of a complete characterization of $M(D_\alpha, D_\alpha)$ for $0 < \alpha \leqq 1$ and H one-dimensional.

References

1. G. H. Hardy, *Divergent series*, Oxford, at the Clarendon, 1949.

2. E. Hille, and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

3. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Engelwood Cliffs, N. J., 1962.

4. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, New York, 1959.

University of Arizona,
Tucson, Arizona